Convolution and convolution-root properties of long-tailed distributions

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Abstract

We obtain a number of new general properties, related to the closedness of the class of long-tailed distributions under convolutions, that are of interest themselves and may be applied in many models that deal with "plus" and/or "max" operations on heavy-tailed random variables. We analyse the closedness property under convolution roots for these distributions. Namely, we introduce two classes of heavy-tailed distributions that are not long-tailed and study their properties. These examples help to provide further insights and, in particular, to show that the properties to be both long-tailed and so-called "generalised subexponential" are not preserved under the convolution roots. This leads to a negative answer to a conjecture of Embrechts and Goldie [10, 12] for the class of long-tailed and generalised subexponential distributions. In particular, our examples show that the following is possible: an infinitely divisible distribution belongs to both classes, while its Lévy measure is neither long-tailed nor generalised subexponential.

Keywords: long-tailed distribution; generalised subexponential distribution; closedness; convolution; convolution root; random sum; infinitely divisible distribution; Lévy measure

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1 Introduction

We assume F to be a distribution on the real line, with the (right) tail distribution function $\overline{F}(x) = 1 - F(x)$. The notation $F_1 * F_2$ is reserved for the convolution of two distributions F_1 and F_2; further $F^{*n} = F * \ldots * F$ denotes the n-fold convolution of F with itself for $n \geq 2$, and $F^{*1} = F$ and F^{*0} denotes the distribution degenerate at zero. All limits are taken as x tends to infinity. For two positive functions f and g, the notation $f(x) \sim g(x)$ means that $\lim f(x)/g(x) = 1$; the notation $f(x) = o(g(x))$ means that $\lim f(x)/g(x) = 0$; and $f(x) = O(g(x))$ means that $\limsup f(x)/g(x) < \infty$. The indicator function $\mathbf{I}(A)$ of an event A takes the value 1 if the event occurs and the value 0 otherwise.

Recall that a distribution F on the real line is *heavy-tailed* if $\int_0^\infty e^{\beta y} F(dy) = \infty$ for all $\beta > 0$, otherwise F is *light-tailed*. A distribution F is *long-tailed*, denoted by $F \in \mathcal{L}$, if $\overline{F}(x+1) \sim \overline{F}(x)$.

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A distribution F *on the positive half-line* is *subexponential*, denoted by $F \in \mathcal{S}$, if $\overline{F^{*2}}(x) \sim 2\overline{F}(x)$. A distribution F *on the whole real line* is subexponential if the distribution F_+ is subexponential, where $F_+(x) = F(x) \cdot \mathbf{I}(x \geq 0)$ for all x, or, equivalently, if $F \in \mathcal{L}$ and if $\overline{F^{*2}}(x) \sim 2\overline{F}(x)$. Note that both subexponentiality and long-tailedness are the *tail properties*: if a distribution F has such a property and $\overline{F}(x) \sim \overline{G}(x)$, then G also has this property. It is known that any subexponential distribution is long-tailed and any long-tailed distribution is heavy-tailed.

More generally, let $\gamma \geq 0$ be fixed. A distribution F on the whole real line *belongs to the distribution class* $\mathcal{L}(\gamma)$ if, for any fixed $c > 0$,

$$\overline{F}(x-c) \sim \overline{F}(x)e^{\gamma c}.$$

A distribution F *belongs to the class* $\mathcal{S}(\gamma)$ if $\int_0^\infty e^{\gamma y}F(dy) < \infty,\ F \in \mathcal{L}(\gamma)$ and if

$$\overline{F^{*2}}(x) \sim 2\overline{F}(x)\int_{-\infty}^{\infty} e^{\gamma y}F(dy).$$

In particular, $\mathcal{L} = \mathcal{L}(0)$ and $\mathcal{S} = \mathcal{S}(0)$. Clearly, distributions from the class $\mathcal{L}(\gamma)$ are light-tailed if $\gamma > 0$. For all $\gamma \geq 0$, the class $\mathcal{L}(\gamma) \setminus \mathcal{S}(\gamma)$ is non-empty, see, e.g., Pitman [27], Leslie [22], Murphree [25], Klüppelberg and Villasenor [21] and Lin and Wang [23] for examples and further analysis. Some systems research and application on the above mentioned distribution classes, please refer to Embrechts et al. [14], Asmussen and Albrecher [2], Foss et al. [19], and so on.

Recall that the classes $\mathcal{S}$ and $\mathcal{L}$ were introduced by Chistyakov [4] and, for $\gamma > 0$, the class $\mathcal{S}(\gamma)$ of distributions supported by the positive half-line was introduced and analysed by Chover et al. [5, 6]. The class $\mathcal{L}$ is closely linked to slow variation ($F \in \mathcal{L}$ iff $\overline{F}(\log x)$ is slowly varying). For $\gamma > 0$, the class $\mathcal{L}(\gamma)$ was introduced by Embrechts and Goldie [10] and is linked to regular variation.

It is known that if $F \in \mathcal{L}$ (or if $F \in \mathcal{S}$), then $F^{*n} \in \mathcal{L}$ (correspondingly $F^{*n} \in \mathcal{S}$), for any $n \geq 2$. These results continue to hold when F^{*n} is replaced by the *compound distribution* $\sum_{n=0}^\infty p_n F^{*n}$ where $0 \leq p_n \leq 1$ for $n = 0, 1, \ldots$, $p_0 < 1$, $\sum_{n=0}^\infty p_n = 1$, given than p_n decay to zero sufficiently fast as $n \to \infty$. In the case of subexponential distributions this is a classical result (based on "Kesten's lemma"; see also [8] and the references therein for modern results in this direction), while the result for long-tailed distributions is quite recent ([1, 24]). Similar results hold for the class $\mathcal{S}(\gamma)$ for any $\gamma > 0$. Therefore, we may say that all these distribution classes are *closed under convolution*.

Embrechts et al. [13] (see also [11]) proved the converse result for subexponential distributions: if $F^{*n} \in \mathcal{S}$ for some $n \geq 2$, then $F \in \mathcal{S}$ (and, in turn, $F^{*m} \in \mathcal{S}$, for all $m \geq 2$). They also proved an analogous result related to the compound distribution, and then similar results for the class $\mathcal{S}(\gamma)$ for any $\gamma > 0$. In short, one can say that, for any $\gamma \geq 0$, the class $\mathcal{S}(\gamma)$ is *closed under convolution roots*.

Embrechts and Goldie (see [10], page 245 and [12], page 270) formulated the conjecture that a similar converse result may hold for long-tailed distributions and, more generally, for any class $\mathcal{L}(\gamma)$, $\gamma \geq 0$.

Conjecture 1 Let $\gamma \geq 0$. If there is $n \geq 2$ such that $F^{*n} \in \mathcal{L}(\gamma)$, then also $F \in \mathcal{L}(\gamma)$.

The following two closely related conjectures may be viewed as natural extensions of Conjecture 1 onto compound distributions and infinitely divisible distributions.

Conjecture 2. Let $\gamma \geq 0$ and $\sum_{n=0}^\infty p_n = 1$, with $p_n \geq 0$ for all n and $p_0 + p_1 < 1$. If a compound distribution $\sum_{n=0}^\infty p_n F^{*n}$ belongs to the class $\mathcal{L}(\gamma)$, then also $F \in \mathcal{L}(\gamma)$.

Conjecture 3. Let $\gamma \geq 0$. If an infinitely divisible distribution H belongs to the class $\mathcal{L}(\gamma)$, then the distribution generated by its Lévy spectral measure belongs to the class $\mathcal{L}(\gamma)$ too.

In this paper, we restrict our attention to the study of the class of long-tailed distributions, and also of its subclass consisting of the so-called *generalised subexponential* distributions.

A distribution F is *generalised subexponential*, denoted by $F \in \mathcal{OS}$, if $\overline{F}(x) > 0$ for all x and if

$$C^*(F) = \limsup \overline{F^{*2}}(x)/\overline{F}(x) < \infty.$$

Note that (a) for any heavy-tailed distribution F on the whole real line, $C^*(F) \geq 2$ (see Theorem 1.2 in [34] for this and further results); (b) clearly, $C^*(F) \geq \liminf \overline{F^{*2}}(x)/\overline{F}(x) \geq 2$ for any distribution on the positive half-line.

The class $\mathcal{OS}$ was first introduced by Klüppelberg [20] for distributions on the positive half-line and was called "weakly idempotent". Later Shimura and Watanabe [29] called it "O-subexponential", or "generalised subexponential", by analogy to"O-regularly varying" in the terminology of Bingham et al. [3]. The definition of the class $\mathcal{OS}$ was extended in [31] to the whole real line.

In this paper, we prove a number of novel properties of long-tailed distributions (see Theorem 2.1) that, in particular, allow us to provide a number of counter-examples to Conjectures 1-3 (see Theorem 2.2 and Proposition 2.1) where the class $\mathcal{L}$ is replaced by the class $\mathcal{L} \cap \mathcal{OS}$. We also provide a simple sufficient condition for the equivalence "$F \in \mathcal{L}$ if and only if $F^{*2} \in \mathcal{L}$" to hold, see Proposition 2.2. Similar problems for light-tailed distributions (with counterexamples to Conjectures 2-3) will be analysed in a companion paper.

The remainder of this paper is organised as follows. In Section 2 we formulate and discuss our main results and their corollaries. In Section 3 we prove Theorem 2.1 and Corollary 2.1. The proofs of Theorem 2.2, Proposition 2.2 and Lemma 2.1 are given in Section 4. Finally, the appendix includes comments related to the condition (2.13) and a sketch of the proof of Proposition 2.1.

2 Main results and related discussions

To formulate our first result, we need further notation. For a distribution F and any constants $a \leq b$, we let $F(a, b] = F(b) - F(a) = \overline{F}(a) - \overline{F}(b)$. Let $X_1, X_2, \ldots$ be independent (not necessarily identically distributed) random variables with corresponding distributions $F_1, F_2, \ldots$. For $n = 0, 1, \ldots$, let $S_n = \sum_{i=1}^n X_i$ be the partial sum with distribution $H_n = F_1 * \cdots * F_n$, where H_0 degenerates at 0. Let τ be an independent counting random variable with distribution function $G(x) = \sum_{n \leq x} p_n$ where $p_n = \mathbf{P}(\tau = n)$, $n = 0, 1, \ldots$. We denote by H_τ the distribution of the random sum $S_\tau = \sum_{i=1}^\tau X_i$. Clearly, $H_\tau = \sum_{n=0}^\infty p_n H_n$. In the particular case where $\{X_i, i \geq 1\}$ are i.i.d. with common distribution F, we have $H_n = F^{*n}$ for $n = 0, 1, \ldots$ and we also use notation $F^{*\tau}$ for $H_\tau = \sum_{n=0}^\infty p_n F^{*n}$.

Theorem 2.1. *(1) Let $n \geq 2$.*
(1a) If $H_n \in \mathcal{L}$, then

$$F_i(x - c, x + c] = o(\overline{H_n}(x)) \quad \text{and} \quad H_i(x - c, x + c] = o(\overline{H_n}(x)), \tag{2.1}$$

for any $c > 0$ and all $i = 1, \ldots, n$.
(1b) Assume $H_m \in \mathcal{L}$ for some $1 \leq m \leq n$ and

$$F_i(x - c, x + c] = o(\overline{H_m}(x)), \tag{2.2}$$

for some $c > 0$ and all $i = m+1, \ldots, n$. Then $H_n \in \mathcal{L}$.
(2) Let τ be an independent counting random variable with bounded support: $\sum_{k=0}^{n} p_k = 1$ and $p_n > 0$, for some $n \geq 1$. Then $H_\tau \in \mathcal{L}$ if and only if $H_n \in \mathcal{L}$.
(3) Assume that $\mathbf{P}(\tau \geq n) > 0$ for some $n \geq 1$ and $H_k \in \mathcal{L}$ for all $k \geq n$. Assume further that there exists a positive constant C such that, for every $n = 1, 2, \ldots$, the following concentration inequality holds:

$$\sup_x H_n(x-1, x] \leq C/\sqrt{n} \tag{2.3}$$

and that, for any $\varepsilon > 0$, there exists $x_0 > 1$ such that, for all $k \geq n$,

$$\sup_{x \geq k(x_0-1)+x_0} \overline{H}_k(x-1)/\overline{H}_k(x) \leq 1 + \varepsilon. \tag{2.4}$$

If, in addition, for any $a > 0$,

$$\overline{G}(ax) = o\left(x^{1/2}\overline{H_n}(x)\right), \tag{2.5}$$

then $H_\tau \in \mathcal{L}$.
(4) Let F_1, F_2 and L_2 be three distributions such that $\overline{F}_2(x) \sim \overline{L_2}(x)$ and $F_1 * F_2 \in \mathcal{L}$. Then $\overline{F_1 * L_2}(x) \sim \overline{F_1 * F_2}(x)$ and, therefore, $F_1 * L_2 \in \mathcal{L}$.

Remark 2.1. *Statement (1b) of Theorem 2.1 is equivalent to the following:*
Assume $F_n \in \mathcal{L}$ for some $n \geq 2$ and

$$F_i(x-c, x+c] = o\big(\overline{F_n}(x)\big),$$

for some $c > 0$ and all $i = 1, \ldots, n$. Then $H_n \in \mathcal{L}$.

Remark 2.2. *Condition (2.3) is very general. It holds if random variables X_i, $i \geq 1$, are i.i.d. with any non-degenerate distribution (see, e.g., [26], Theorem 2.22). More generally, (2.3) holds if random variables X_i are assumed to be independent, but not necessarily identically distributed, and there exists $c > 0$ such that*

$$\inf_{i \geq 1} \mathbf{P}(X_i \in [-c, c]) > 0 \quad \text{and} \quad \inf_{i \geq 1} \mathbf{V}ar(X_i \mid X_i \in [-c, c]) > 0, \tag{2.6}$$

see e.g. [17], Lemma 4.1. Moreover, it is enough to assume that (2.6) holds only for a positive proportion of the summands: if c_n is the number of X_i, $i \leq n$ that satisfy (2.6), then $c_n/n \geq c > 0$ for some $c > 0$ and for all sufficiently large n.

Some other conditions for the concentration inequality can be found in theorems that precede Theorem 2.22 of the book [26] (e.g., Theorems 2.17 and 2.18).

In the case of i.i.d. summands, Theorem 2.1 leads to the following corollary.

Corollary 2.1. *(1) Assume a distribution F to be such that $F^{*n} \in \mathcal{L}$, for some $n \geq 1$. Then $F^{*k} \in \mathcal{L}$, for all $k \geq n$.*
(2) Let τ be a counting random variable with bounded support: $\sum_{k=0}^{n} p_k = 1$ and $p_n > 0$, for some $n \geq 1$. Then $F^{*\tau} \in \mathcal{L}$ if and only if $F^{*n} \in \mathcal{L}$.
(3) If, for some $n \geq 1$, $\mathbf{P}(\tau \geq n) > 0$ and $F^{*n} \in \mathcal{L}$, and if

$$\overline{G}(ax) = o\left(x^{1/2}\overline{F^{*n}}(x)\right) \tag{2.7}$$

for any $a > 0$, then $F^{*\tau} \in \mathcal{L}$.
(4) Let F and L be two distributions such that $F^{*2} \in \mathcal{L}$ and $\overline{F}(x) \sim \overline{L}(x)$. Then $\overline{L^{*2}}(x) \sim \overline{F^{*2}}(x)$ and, therefore, $L^{*2} \in \mathcal{L}$.

In order to illustrate the above results and to formulate the new ones, we need further notion and notation. Recall that a distribution F is *dominatedly-varying-tailed*, denoted by $F \in \mathcal{D}$, if for some (or equivalently, for all) $c \in (0,1)$,

$$\limsup \overline{F}(cx)/\overline{F}(x) < \infty.$$

A distribution F belongs to the *generalised long-tailed* distribution class $\mathcal{OL}$, if $\overline{F}(x) > 0$ for all x and if, for any $c > 0$,

$$C(F,c) = \limsup \overline{F}(x-c)/\overline{F}(x) < \infty.$$

The class $\mathcal{OL}$ is significantly broader than the class $\mathcal{L}$ and, in particular, the class $\mathcal{OL}$ covers all classes $\mathcal{L}(\gamma)$, $\gamma \geq 0$.

The classes $\mathcal{D}$ and $\mathcal{OL}$ were introduced by [15] and [29], respectively. Note that $\mathcal{OS}$ is a proper subclass of the class $\mathcal{OL}$, see e.g. [29] or [31].

Remark 2.3. *Statement (1a) of Theorem 2.1 is quite general – in particular, it may be applied in the case where $n = 2$, and F_1 is not long-tailed itself. We present two examples in the Appendix below. In Example 5.1, there are two distributions $F_1 \in \mathcal{OL}$ and F_2 such that $\overline{F_2}(x) = o(\overline{F_1}(x))$; and in Example 5.2, there are two distributions $F_1 \notin \mathcal{OL}$ and F_2 such that $\liminf \overline{F_2}(x)/\overline{F_1}(x) = 0$ and $\limsup \overline{F_2}(x)/\overline{F_1}(x) = \infty$. In both examples, $F_1 \notin \mathcal{L} \cup \mathcal{D}$ and $F_1 * F_2 \in \mathcal{L}$.*

Remark 2.4. *In Corollary 2.1, parts (1) and (3), if $n \geq 2$, then distributions F^{*k} may be not long-tailed for $1 \leq k \leq n-1$, in general – see, e.g., families of distributions $\mathcal{F}_i(0), i = 1,2$ that are introduced below. Therefore this result is a reasonable generalisation of Theorem 6 of Leipus and Šiaulys [24]. Also, Leipus and Šiaulys [24] require condition* (2.7) *with $n = 1$ that is stronger than our condition if F does not belong to the class $\mathcal{OS}$.*

Remark 2.5. *The results of part (1) of Theorem 2.1 may be generalised onto the case of weakly dependent random variables. Here is an example for $n = 2$, with a particular choice of a weak dependence structure of random variables. Let X_i be a random variable with the distribution F_i supported on whole real line, $i = 1,2$. Assume that a random vector (X_1, X_2) has the two-dimensional Farlie-Gumbel-Morgenstern (FGM) joint distribution:*

$$\mathbf{P}\Big(\bigcap_{i=1}^{2}\{X_i \leq x_i\}\Big) = \prod_{i=1}^{2} F_i(x_i)(1 + \theta_{12}\overline{F_1}(x_1)\overline{F_2}(x_2)), \tag{2.8}$$

where $\theta_{12} \neq 0$ is a constant such that $a =\mid \theta_{12} \mid \leq 1$.

For any $0 < T_i \leq \infty, i = 1,2,$ direct calculations show that

$$\mathbf{P}\Big(\bigcap_{i=1}^{2}\{X_i \in (x_i, x_i+T_i]\}\Big) = \prod_{i=1}^{2} F_i(x_i, x_i+T_i]\big(1 + \theta_{12}\prod_{i=1}^{2}(1 - \overline{F_i}(x_i+T_i) - \overline{F_i}(x_i))\big). \tag{2.9}$$

Then, by (2.9)*, we have for $1 \leq i \neq j \leq 2$ and all x_i, x_j,*

$$\begin{aligned}\mathbf{P}(X_i \in (x_i, x_i+T_i] | X_j = x_j) = F_i(x_i, x_i+T_i] \\ \cdot\big(1 + \theta_{12}(1 - \overline{F_i}(x_i+T_i) - \overline{F_i}(x_i))(1 - 2\overline{F_j}(x_j))\big).\end{aligned} \tag{2.10}$$

Take $a < 1$. One may show that the statements (1a) and (1b) of Theorem 2.1 still hold under new assumptions by simply following their proofs, with a suitable use of equalities (2.8) *and* (2.10).

Now we discuss the closedness property under convolution roots related to the class $\mathcal{L} \cap \mathcal{OS}$. We show that all three Conjectures 1-3 do not take place in the class $\mathcal{L} \cap \mathcal{OS}$. We provide precise examples and the intuition behind. All our examples involve absolutely continuous distributions. In more detail, we introduce below two families of distributions, $\mathcal{F}_1(0)$ and $\mathcal{F}_2(0)$, that have different properties and are built up around random variables of the form

$$\xi = \eta(1+U) \tag{2.11}$$

where η has a discrete and heavy-tailed distribution and U is an independent random variable with a smooth distribution with bounded support. For simplicity, we assume U to be uniformly distributed, but its distribution may be taken from a larger class. Further, classes $\mathcal{F}_1(0)$ and $\mathcal{F}_2(0)$ may be extended, thanks to part (4) of Corollary 2.1 on the tail-equivalence.

Definition 2.1. *Class $\mathcal{F}_1(0)$ is a 4-parametric family of distributions $F = F(\alpha, b, t, A)$ of random variables*

$$\xi = \eta(1+U^{1/b})^t \tag{2.12}$$

with density $f = f(\alpha, b, t, A)$. Here $\alpha \in [1/2, 1)$, $b > 0$ and $t \geq 1$ are constants. Further, η is a discrete random variable with distribution $\mathbf{P}(\eta = a_n) = Ca_n^{-\alpha}$, where $C = (\sum_{n=0}^{\infty} a_n^{-\alpha})^{-1}$ is the normalising constant and a sequence $A = \{a_n\}$ is defined as follows. Let $r = 1 + 1/\alpha > 2$ and a constant $a > 1$ be so large that $a^r > 2^{t+2}a$, then $a_n = a^{r^n}$ for $n = 0, 1, \ldots$. Finally, U is a random variable having uniform distribution in the interval $(0,1)$, and U and η are mutually independent.

A number of "good" properties of the class $\mathcal{F}_1(0)$ is given in the following theorem. In particular, the theorem provides a negative answer to Conjectures 1-3 related to the class $\mathcal{L} \cap \mathcal{OS}$.

Theorem 2.2. *For any distribution $F \in \mathcal{F}_1(0)$, the following conclusions hold.*
(1) F is neither long-tailed nor generalised subexponential, while $F \in \mathcal{OL}$ and $F^{*n} \in \mathcal{L} \cap \mathcal{OS} \setminus \mathcal{S}$, for all $n \geq 2$.
(2) $F^{*\tau} \in \mathcal{L} \setminus \mathcal{S}$, for any counting random variable τ with distribution G such that $\mathbf{P}(\tau \geq 2) > 0$ and for any $a > 0$

$$\overline{G}(ax) = o\big(x^{1/2}\overline{F^{*2}}(x)\big). \tag{2.13}$$

(2a) Further, if condition (2.13) is replaced by the following: for any $0 < \varepsilon < 1$, there is an integer $M = M(\varepsilon) \geq 2$ large enough such that

$$\sum_{n=M}^{\infty} p_n \overline{F^{*n}}(x) \leq \varepsilon \overline{F^{*\tau}}(x), \;\; for \;\; all \;\; x \geq 0, \tag{2.14}$$

then $F^{*\tau} \in \mathcal{L} \cap \mathcal{OS}$.
(2b) Assume now that $\mathbf{E}\tau < \infty$. Then (2.14) implies that

$$\liminf \overline{F^{*\tau}}(x)/\overline{F}(x) = \mathbf{E}\tau \geq 2 \liminf \overline{F^{*\tau}}(x)/\overline{F^{*2}}(x) \geq 2\sum_{m=1}^{\infty}(p_{2m} + p_{2m+1})m. \tag{2.15}$$

Further, if condition

$$\sum_{m=1}^{\infty}\Big(\sum_{k=2(m-1)+1}^{2m} p_k\Big)\big(C^*(F^{*2}) - 1 + \varepsilon_0\big)^m < \infty \tag{2.16}$$

holds for some $\varepsilon_0 > 0$, then

$$2 \limsup \overline{F^{*\tau}}(x)/\overline{F^{*2}}(x) \le 2 \sum_{m=1}^{\infty} m(p_{2m-1} + p_{2m})(C^*(F^{*2}) - 1)^{m-1} < \infty \tag{2.17}$$

while $\limsup \overline{F^{*\tau}}(x)/\overline{F}(x) = \infty$.

(3) For any distribution $F \in \mathcal{F}_1(0)$, there is an infinitely divisible distribution H such that F is generated by its Lévy measure and the following holds: $H \in (\mathcal{L} \cap \mathcal{OS}) \setminus \mathcal{S}$, while F is neither long-tailed nor generalised subexponential.

Remark 2.6. *Assume a random variable τ has a Poisson distribution with parameter $\mu = \mathbf{E}\tau$. Let $r = (C^*(F^{*2}) - 1)^{1/2}$. Direct computations show that the lower bound in* (2.15) *is equal to*

$$\mu + (1 - e^{-2\mu})/2,$$

and the upper bound in (2.17) *is equal to*

$$\frac{\mu + 1}{2r}\left(e^{\mu(r-1)} - e^{-\mu(r+1)}\right) + \frac{\mu}{2}\left(e^{\mu(r-1)} + e^{-\mu(r+1)}\right).$$

The same (lower and upper) bounds hold for the lower and upper limits of $2\overline{H}(x)/\overline{F^{*2}}(x)$ in part (3) of Theorem 2.2. In this case, μ is precisely given in the proof, see Section 3.

Lemma 2.1. *The following condition implies* (2.15)*: there exist $n \ge 1$ and $\varepsilon_0 > 0$ such that*

$$\sum_{m=1}^{\infty} \Big(\sum_{k=(m-1)n+1}^{mn} p_k \Big) \Big(C^*(F^{*n}) - 1 + \varepsilon_0\Big)^m < \infty. \tag{2.18}$$

One can see that condition (2.16) is a particular case of condition (2.18), with $n = 2$.

Remark 2.7. *Condition* (2.18) *holds if a distribution G is either Poisson ($p_k = \lambda^k e^{-\lambda}/(k!)$, $k = 0, 1, \ldots$) or Geometric ($p_k = qp^k$, $k = 0, 1, \ldots$, with $p < 1/(C^*(F^{*n}) - 1 + \varepsilon_0)$, for some $\varepsilon_0 > 0$). Note that* (2.18) *is a natural generalisation of the classical sufficient condition for subexponentiality of a random sum (where $n = 1$ and $C^*(F) = 2$), see e.g. Theorem 4 in [5] or Theorem 3 and its Remark in [13]. Clearly, a distribution G satisfying* (2.18) *is light-tailed.*

Here is an example of a heavy-tailed distribution G that satisfies condition (2.14).

Example 2.1. *Let $n = 1$. Assume $F \in \mathcal{D}$, then by Theorem 3 of Daley et al. [9], there are two positive constants C and α such that*

$$\sup_{x \ge 0} \overline{F^{*k}}(x)/\overline{F}(x) \le Ck^{\alpha}, \ \textit{for all } k \ge 1. \tag{2.19}$$

Take a counting random variable τ with distribution G given by $\mathbf{P}(\tau = k) = p_k = Kk^{-\beta}$ for some $\beta > \alpha + 2$, where $K = \left(\sum_{k=1}^{\infty} k^{-\beta}\right)^{-1}$ is the normalising constant. Clearly, condition (2.14) takes place and G is a heavy-tailed distribution. However, condition (2.18) in Remark 2.7 does not hold.

Remark 2.8. *Note that all distributions F considered in Theorem 2.2 are generalised long-tailed, that is $\mathcal{F}_1(0) \subset \mathcal{OL}$. One may guess that such a condition may be essential for F^{*2} to be long-tailed. However, this is not the case: we introduce below another family $\mathcal{F}_2(0)$ of heavy-tailed distributions F such that $F \notin \mathcal{OL}$ while $F^{*2} \in \mathcal{L}$ and, moreover, $F^{*2} \in \mathcal{OS}$.*

Definition 2.2. *Class $\mathcal{F}_2(0)$ is a 3-parametric family of heavy-tailed distributions $F = F(\alpha, t, A)$ of random variables*

$$\xi = \eta^{1/t}(1+U)^{1/t} \tag{2.20}$$

with density $f = f(\alpha, t, A)$. Here $t \in (1,2)$, $\alpha \in ((1-t)/t, 1/t)$ and the sequence $A = \{a_n\}$ and random variables η and U are defined as in Definition 2.1.

Properties of the class $\mathcal{F}_2(0)$ are summarised in the following proposition.

Proposition 2.1. *Let $F \in \mathcal{F}_2(0)$, then $F^{*n} \in \mathcal{L} \setminus \mathcal{S}$, for all $n \geq 2$. Further, for any $n \geq 2$, $F^{*n} \in \mathcal{OS}$ when $\alpha \in [1/2, 1/t)$ and $F^{*n} \notin \mathcal{OS}$ when $\alpha \in ((t-1)/t, 1/2)$, while $F \notin \mathcal{OL}$, and therefore $F \notin \mathcal{L} \cup \mathcal{D}$.*

Remark 2.9. *In addition, for the class $\mathcal{F}_2(0)$ with $\alpha \in [1/2, 1/t)$, the natural analogues of statements (2) and (3) of Theorem 2.2 do hold.*

The **proof of Proposition 2.1** is quite similar to that of Theorem 2.2. For the sake of completeness, we decided to give it in Subsection 4.2 of Appendix.

Theorem 2.2 and Proposition 2.1 provide a good number of new examples of distributions from the classes $\mathcal{L} \setminus \mathcal{S}$ and $(\mathcal{L} \cap \mathcal{OS}) \setminus \mathcal{S}$.

Remark 2.10. *Watanabe and Yamamuro [32] commented in Remark 2.3 that Shimura and Watanabe [30] provided a counter-example to Conjecture 1. Also, [32] pointed out that [30] did not find an answer to the corresponding Conjectures 2-3 related to distributions of random sums (compound distribution or random convolution) and infinitely divisible distribution. In addition, [32] stated that the class $\mathcal{OS}$ is not closed under convolution roots, but we did not find any corresponding result for the intersection of the classes $\mathcal{L} \cap \mathcal{OS}$.*

Recently we were in touch with Dr Shimura who has sent us privately an unpublished English translation of Research Report [30]. We have found that the counter-example there seems to be correct, but is described implicitly, so it is difficult to follow. Also, the example relates to a distribution that is neither absolutely continuous nor discrete.

Finally, we show that the long-tailedness property is preserved under convolution roots within the class $\mathcal{OS}$. Namely, the following result holds.

Proposition 2.2. *If $F \in \mathcal{OS}$, then $F \in \mathcal{L}$ if and only if $F^{*2} \in \mathcal{L}$.*

3 Proofs of Theorem 2.1 and Corollary 2.1

In order to prove Theorem 2.1, we first recall a number of known properties of long-tailed distributions. We consider here distributions on the whole real line.

The definition of the class $\mathcal{L}$ and the diagonal argument lead to the following result.

Property 1. *Distribution F is long-tailed if and only if there exists a monotone increasing function $h(x) \uparrow \infty$ such that $h(x) < x$ and $F(x-h(x), x+h(x)] = o(\overline{F}(x))$ (then we say that $\overline{F}$ is h-insensitive).*

See, e.g., [19], Chapter 2 for Property 1 and for h-insensitivity and other properties of class $\mathcal{L}$. Further, [10] and [12] show that the class $\mathcal{L}$ is closed under convolution and mixture.

Property 2. *Let F_1 and F_2 be two distributions.*
(1) Assume $F_1 \in \mathcal{L}$. Then $F_1 * F_2 \in \mathcal{L}$ if either (a) $F_2 \in \mathcal{L}$ or (b) $\overline{F_2}(x) = o(\overline{F_1}(x))$. In the latter case, $\overline{F}_1(x) \sim \overline{F_1 * F_2}(x)$.
(2) If $F_1, F_2 \in \mathcal{L}$, then $pF_1 + (1-p)F_2 \in \mathcal{L}$, for any $p \in [0,1]$.

Albin [1] and then Leipus and Šiaulys [24] extended Property 2 (1) onto random convolutions.

Property 3. *If $F \in \mathcal{L}$ and if (2.7) holds for $n = 1$ and for all $a > 0$, then $F^\tau \in \mathcal{L}$.*

We proceed now with the **Proof of Theorem 2.1.**

Proof of (1a). First, we prove (2.1) for $i = 1$. By $H_n \in \mathcal{L}$, we may choose $h(x) \uparrow \infty$ such that $\overline{H}_n$ is nh-insensitive. Then, by Property 1,

$$\mathbf{P}\big(S_n \in (x - nh(x), x + nh(x)]\big) = o\big(\overline{H_n}(x)\big).$$

Note that

$$\mathbf{P}\big(S_n \in (x - nh(x), x + nh(x)]\big) \geq \mathbf{P}\big(X_1 \in (x - h(x), x + h(x)]\big) \cdot \prod_{j=2}^{n} \mathbf{P}\big(X_j \in (-h(x), h(x)]\big)$$

and

$$\mathbf{P}\big(-h(x) < X_j \leq h(x)\big) \to 1, \quad j = 2, \ldots, n.$$

Then the first part of (2.1) follows. Since $H_1 = F_1$, the second part follows too.

If $i > 1$, then the proof of the first part of (2.1) is the same. For the second part, we may represent S_n as a sum of mutually independent random variables $S_n = S_i + X_{i+1} + \ldots + X_n$ and apply the arguments from above. □

Proof of (1b). It is enough to prove the result for $m = 1$ and $n = 2$, and then use the induction argument. First, by monotonicity of distribution functions and since F_1 is long-tailed, we may obtain that $F_2(x - c, x + c] = o(\overline{F_1}(x))$ for any $c > 0$ and, therefore,

$$\alpha_c(x) =: \sup_{y \geq x} \Big(F_2(x - c, x + c] / \overline{F_1}(y) \Big) \downarrow 0.$$

Then one can use the diagonal argument to conclude that there exists a positive function $h_1(x) \uparrow \infty$ such that

$$F_2(x - 2h_1(x), x + 2h_1(x)] = o(\overline{F_1}(x)). \tag{3.1}$$

Further, since F_1 is long-tailed, one can find a function $h_2(x) \uparrow \infty$ such that $\overline{F_1}$ is $2h_2$-insensitive. Let $h(x) = \min(h_1(x), h_2(x))$. Then $\overline{F_1}$ is $2h$-insensitive and (3.1) holds with h in place of h_1.

Let X_1, X_2 be two independent random variables where X_1 has distribution F_1 and X_2 has distribution F_2. Then, for any $c > 0$ and for x such that $h(x) > c$,

$$\begin{aligned} & F_1 * F_2(x - c, x + c] = \mathbf{P}(X_1 + X_2 \in (x - c, x + c]) \\ \leq \;& F_2(x - 2h(x), x] + \Big(\int_{-\infty}^{x - 2h(x)} + \int_x^\infty \Big) F_2(dy) F_1(x - y - c, x - y + c]. \end{aligned}$$

There are three terms on the right-hand side. The first term is $o(\overline{F_1}(x))$, by condition (3.1). It is also $o(\overline{F_1 * F_2}(x))$ since $\overline{F_1 * F_2}(x) \geq \overline{F_1}(x_0)\overline{F}(x - x_0) \sim \overline{F_1}(x_0)\overline{F_1}(x)$, where x_0 is any number such that $\overline{F_2}(x_0) > 0$.

Then the second term is not bigger than

$$\alpha_c(2h(x)-c)\int_{-\infty}^{x-2h(x)} F_2(dy)\overline{F_1(x-y)} \le \alpha_c(2h(x)-c)\overline{F_1*F_2}(x) = o(\overline{F_1*F_2}(x)).$$

Finally, the last term is not bigger than

$$\begin{aligned}
& \sum_{k=0}^{\infty} F_2(x+kc, x+(k+1)c]F_1(-(k+2)c, -(k-1)c] \\
\le\ & 3\sup_{y\ge x} F_2(y, y+c]F_1(c) \\
=\ & o(\overline{F_1}(x)) = o(\overline{F_1*F_2}(x)).
\end{aligned}$$

Thus $F_1 * F_2 \in \mathcal{L}$. □

Proof of (2). Assume first that $H_n \in \mathcal{L}$. Then, by property (1a), $H_k(x-c, x+c] = o(\overline{H_n}(x))$, for all $k = 1, \dots, n$ and for any fixed $c > 0$. Then

$$H_\tau(x-c, x+c] = \sum_{k=1}^{n} p_k H_k(x-c, x+c] = o(\overline{H_n}(x)),$$

and $H_\tau \in \mathcal{L}$ follows.

Vice versa, if $H_\tau \in \mathcal{L}$, then

$$H_k(x-c, x+c] \le H_\tau(x-c, x+c])/p_k = o(\overline{H}_\tau(x))$$

for each k such that $p_k > 0$ and, in particular, for $k = n$. Let $x_1, \dots, x_n$ be positive numbers such that $\overline{F}_i(x_i) > 0$. Clearly, $\overline{H}_n(x) \ge \overline{H_k}(x - \sum_{i=k+1}^n x_i)\prod_{i=k+1}^n \overline{F}_i(x_i)$, for all $k = 1, \dots, n-1$ and then

$$\overline{H}_\tau(x) \le \sum_{k=1}^{n} p_k \overline{H_k}\Big(x - \sum_{i=k+1}^{n} x_i\Big) \le \overline{H_n}(x)/\Big(\prod_{i=1}^{n}\overline{F}_i(x_i)\Big).$$

Thus $H_n \in \mathcal{L}$ follows from

$$H_n(x-c, x+c] = o(\overline{H}_\tau(x)) = o(\overline{H_n}(x)).$$

□

Proof of (3). We may assume, without loss of generality, that $p_n = \mathbf{P}(\tau = n) > 0$. Further, we may assume that $\mathbf{P}(\tau > n) > 0$ – otherwise the result follows from the previous statement.

Let $P_n = \mathbf{P}(\tau \le n) = \sum_{k=0}^n p_k$ and $Q_n = \mathbf{P}(\tau > n) = \sum_{k=n+1}^\infty p_k$. Further, let $H^{(1)}(x) = \sum_{k=1}^n p_k H_k(x)/P_n$ and $H^{(2)}(x) = \sum_{k=n+1}^\infty p_k H_k(x)/Q_n$. Since $H = P_n H^{(1)} + Q_n H^{(2)}$, it is enough to show that both $H^{(1)}$ and $H^{(2)}$ are long-tailed – see Property 2 (2). By the previous statement (2), we have $H^{(1)} \in \mathcal{L}$. Then the argument from [24] implies

$$\begin{aligned}
\overline{H^{(2)}}(x-1) &= \sum_{n+1\le k\le (x-x_0)/(x_0-1)} p_k\overline{H_k}(x-1)/Q_n + \sum_{k>(x-x_0)/(x_0-1)} p_k\overline{H_k}(x-1)/Q_n \\
&\le (1+\varepsilon)\overline{H^{(2)}}(x) + \sum_{k>(x-x_0)/(x_0-1)} p_k H_k((x-1, x])/Q_n \\
&\le (1+\varepsilon)\overline{H^{(2)}}(x) + \frac{\overline{G}((x-x_0)/(x_0-1))}{Q_n\sqrt{(x-x_0)/(x_0-1)}} \\
&= (1+\varepsilon)\overline{H^{(2)}}(x) + o(\overline{H^{(2)}}(x)).
\end{aligned}$$

Since $\varepsilon > 0$ is arbitrary, the distribution $H^{(2)}$ is long-tailed. $\square$

Proof of (4). By part (1) of the theorem, there exists a function $h(x) \uparrow \infty$ such that $\overline{F_1 * F_2}$ is h-insensitive and $\overline{F_1}(x - h(x)) - \overline{F_1}(x) = o(\overline{F_1 * F_2}(x))$. Then

$$\int_{x-h(x)}^{x} F_1(dy)\overline{F}_2(x-y) \le F_1(x-h(x), x] = o(\overline{F_1 * F_2}(x))$$

and, similarly, $\int_{x-h(x)}^{x} F_1(dy)\overline{L_2}(x-y) = o(\overline{F_1 * F_2}(x))$.

Next,

$$\overline{F_1 * F_2}(x)/\overline{F}_1(x) \sim \overline{F_1 * F_2}(x-h(x))/\overline{F}_1(x) \ge \overline{F}_1(x)\overline{F}_2(-h(x))/\overline{F}_1(x) = 1 - o(1)$$

and then

$$\begin{aligned}
\overline{F}_1(x) &\ge \left(\int_{x}^{x+h(x)} + \int_{x+h(x)}^{\infty}\right) F_1(dy)\overline{F}_2(x-y) \\
&\ge o(\overline{F_1 * F_2}(x)) + \int_{x+h(x)}^{\infty} F_1(dy)\overline{F}_2(-h(x)) \\
&= o(\overline{F_1 * F_2}(x)) + (\overline{F}_1(x+h(x)) - \overline{F}_1(x))(1+o(1)) + \overline{F}_1(x)(1+o(1)) \\
&= o(\overline{F_1 * F_2}(x)) + \overline{F}_1(x).
\end{aligned}$$

Therefore, $\int_x^\infty F_1(dy)\overline{F}_2(x-y) = \overline{F}_1(x) + o(\overline{F_1 * F_2}(x))$ and the same holds with L_2 in place of F_2 in the left-hand side of the latter equality.

Further, due to the monotonicity of distribution functions, $\overline{F}_2(x-y) \sim \overline{L_2}(x-y)$ uniformly in $x - y \ge h(x)$. Therefore

$$\int_{-\infty}^{x-h(x)} F_1(dy)\overline{F}_2(x-y) \sim \int_{-\infty}^{x-h(x)} F_1(dy)\overline{L_2}(x-y).$$

Finally,

$$\begin{aligned}
\overline{F_1 * L_2}(x) &= \int_{x}^{\infty} F_1(dy)\overline{L_2}(x-y) + \int_{-\infty}^{x-h(x)} F_1(dy)\overline{L_2}(x-y) + \int_{x-h(x)}^{x} F_1(dy)\overline{L_2}(x-y) \\
&\sim \int_{x}^{\infty} F_1(dy)\overline{F}_2(x-y) + \int_{-\infty}^{x-h(x)} F_1(dy)\overline{F}_2(x-y) + o(\overline{F_1 * F_2}(x)) \\
&\sim \overline{F_1 * F_2}(x),
\end{aligned}$$

and therefore $F_1 * L_2 \in \mathcal{L}$. $\square$

Proof of Corollary 2.1. We need to prove statements (1), (3) and (4) only.

Proof of (1). If $F^{*2} \in \mathcal{L}$, then by statement (1a) of Theorem 2.1, $\overline{F}(x-t) - \overline{F}(x+t) = o(\overline{F^{*2}}(x))$ for any $t > 0$. Further, by statement (1b) of Theorem 2.1, we have $F^{*3} = F^{*2} * F \in \mathcal{L}$. Then Property 2 and the induction argument complete the proof. $\square$

Proof of (3). Condition (2.3) follows from [26], Theorem 2.22. So we have to verify (2.4) only. Due to Lemma 2.1 from [1] or Lemma 4 from [24], for any long-tailed distribution V and for any $\varepsilon > 0$, there is $x_0 > 1$ such that, for all $i \ge 1$,

$$\sup_{x \ge n(x_0-1)+x_0} \overline{V^{*n}}(x-1)/\overline{V^{*n}}(x) \le 1 + \varepsilon. \tag{3.2}$$

Clearly, if there are, say, m long-tailed distributions $V_1, \ldots, V_m$, then (3.2) holds again for some $x_0 > 1$ and for any V_i in place of V. Using similar arguments, one can also show that , for any $i \geq 1$, inequalities (3.2) hold for U_n in place of V^{*n} where U_n is any convolution of n distribution functions taken from the set $\{V_1, \ldots, V_m\}$ – namely, $U_n = V_1^{*j_1} * \ldots * V_m^{*j_m}$ where $j_1 + \ldots + j_m = n$. As the corollary, we may take $m = n$ and then $V_l = F^{*(n+l)}$, for $l = 1, \ldots, n$, to conclude that inequalities (3.2) continue to hold for $i \geq n$, with F^{*n} in place of V^{*n}. □

Proof of (4). We have to apply part (4) of Theorem 2.1 twice, first to move from F^{*2} to $F * L$ and then from $F * L$ to L^{*2}. □

4 Proofs of Theorem 2.2, Proposition 2.2 and Lemma 2.1

We start with a simple auxiliary result.

Lemma 4.1. *Assume that a distribution F is absolutely continuous with density f. If*

$$f(x) = o(\overline{F}(x)) \text{ a.e.,} \tag{4.1}$$

then F is long-tailed. □

Proof. Indeed, let $\varepsilon(x) = \sup_{y \geq x} f(y)/\overline{F}(y)$. Since $\varepsilon(x) \downarrow 0$, we have

$$\overline{F}(x+1) \leq \overline{F}(x) = \overline{F}(x+1) + \int_x^{x+1} f(y)dy \leq \overline{F}(x+1) + \varepsilon(x)\overline{F}(x),$$

and the result follows. □

Proof of Theorem 2.2. Start with **Proof of (1).** Recall that $F \notin \mathcal{L}$ implies with necessity that $F^{*n} \notin \mathcal{S}$ for all $n \geq 2$. Then, by Corollary 2.1 of the present paper and Proposition 2.6 from [25], we only need to prove that $F \notin \mathcal{L} \cup \mathcal{OS}$, $F \in \mathcal{OL}$ and $F^{*2} \in \mathcal{L} \cap \mathcal{OS}$.

First, we find closed-form representations for distribution F and its density f. Clearly, $\eta \leq \xi \leq 2^t \eta$. Since

$$\mathbf{E}\eta^s = C \sum_{n=0}^{\infty} a_n^{s-\alpha} < \infty \tag{4.2}$$

if and only if $s < \alpha$, the same holds for ξ, and distribution F is heavy-tailed with infinite mean. Further, by (2.12) we have, for $n \geq 1$,

$$\begin{aligned}
& F(a_n, x]\mathbf{I}(x \in [a_n, a_{n+1})) \\
= \ & \mathbf{P}(\eta = a_n)\mathbf{P}(1 < (1 + U^{1/b})^t \leq x/a_n)\Big(\mathbf{I}(x \in [a_n, 2^t a_n)) + \mathbf{I}(x \in [2^t a_n, a_{n+1}))\Big) \\
= \ & C a_n^{-\alpha}\Big((a_n^{-1}x)^{1/t} - 1\Big)^b \mathbf{I}(x \in [a_n, 2^t a_n)) + C a_n^{-\alpha}\mathbf{I}(x \in [2^t a_n, a_{n+1})).
\end{aligned}$$

Then

$$f(x) = Cbt^{-1} \sum_{n=0}^{\infty} x^{1/t-1} a_n^{-\alpha-1/t}\Big((x a_n^{-1})^{1/t} - 1\Big)^{b-1} \mathbf{I}(x \in [a_n, 2^t a_n)) \tag{4.3}$$

and

$$\begin{aligned}\overline{F}(x) &= \mathbf{I}(x<a_0)+\sum_{n=0}^{\infty}\Big(\mathbf{P}(\xi\in(a_n,a_{n+1}])-\mathbf{P}(\xi\in(a_n,x])+\mathbf{P}(\xi>a_{n+1})\Big)\mathbf{I}(x\in[a_n,2^ta_n))\\ &\qquad +\sum_{n=0}^{\infty}\Big(\mathbf{P}(\xi\in(2^ta_n,a_{n+1}])-\mathbf{P}(\xi\in(2^ta_n,x])+\mathbf{P}(\xi>a_{n+1})\Big)\mathbf{I}(x\in[2^ta_n,a_{n+1}))\\ &= \mathbf{I}(x<a_0)+\sum_{n=0}^{\infty}\Bigg(\Big(\sum_{i=n}^{\infty}Ca_i^{-\alpha}-Ca_n^{-\alpha}\Big((x/a_n)^{1/t}-1\Big)^b\Big)\mathbf{I}(x\in[a_n,2^ta_n))\\ &\qquad +\sum_{i=n+1}^{\infty}Ca_i^{-\alpha}\mathbf{I}(x\in[2^ta_n,a_{n+1}))\Bigg),\quad x\in(-\infty,\infty).\end{aligned}\tag{4.4}$$

Now, we prove that $F\in\mathcal{OL}\backslash\mathcal{L}$. Note that $a_{n+1}a_n^{-2}\to\infty$ as $n\to\infty$, so for any $K>0$,

$$\sum_{n\geq N}a_n^{-K}\sim a_N^{-K},\quad \overline{F}(a_n)\sim\mathbf{P}(\eta=a_n)\quad\text{ and }\quad\mathbf{P}(\eta>a_n)=o(\mathbf{P}^2(\eta=a_n)).\tag{4.5}$$

From (4.4) and (4.5), we have

$$\overline{F}(2^ta_n)\sim\mathbf{P}(\eta=a_{n+1})=Ca_{n+1}^{-\alpha}=Ca_n^{-1-\alpha}$$

and

$$\overline{F}(2^ta_n-1)-\overline{F}(2^ta_n)=Ca_n^{-\alpha}\left(1-\Big((2^t-a_n^{-1})^{1/t}-1\Big)^b\right)\sim Cbt^{-1}2^{-t+1}a_n^{-\alpha-1},$$

as $n\to\infty$. Therefore

$$\limsup_{x\to\infty}\overline{F}(x-1)/\overline{F}(x)=bt^{-1}2^{-t+1}+1,\tag{4.6}$$

so $F\notin\mathcal{L}$, but $F\in\mathcal{OL}$.

Next, we prove that $F^{*2}\in\mathcal{L}$. Let (η_i,U_i), $i=1,2$ be two independent copies of (η,U), and let $\xi_i=\eta_i(1+U_i^{1/b})^t$ and $S_2=\xi_1+\xi_2$. The random variable S_2 has an absolutely continuous distribution, say, $H=F^{*2}$ with density function

$$h(x)=\int_0^x f(y)f(x-y)dy=2\int_{x/2}^x f(y)f(x-y)dy,\ x\in(-\infty,\infty).\tag{4.7}$$

Clearly, $h(x)>0$ if and only if $a_n+a_0<x<2^{t+1}a_n$, for $n=0,1,\ldots$. According to Lemma 4.1, it is enough to show that

$$h(x)=o(\overline{H}(x)).\tag{4.8}$$

We consider two cases: (i) $x\in J_{n,1}=[a_n+a_0,3\cdot2^{t-1}a_n)$ and (ii) $x\in J_{n,2}=[3\cdot2^{t-1}a_n,2^{t+1}a_n)$ for $n=0,1,\ldots$.

In the case (i), representations (4.3) and (4.7) lead to

$$\begin{aligned}h(x) &\leq 2Cbt^{-1}a_n^{-\alpha-1/t}\int_{a_n}^{2^ta_n}y^{1/t-1}\Big((ya_n^{-1})^{1/t}-1\Big)^{b-1}f(x-y)dy\\ &\leq 2Cbt^{-1}a_n^{-\alpha-1}\int_{a_n}^{2^ta_n}f(x-y)dy\leq 2Cbt^{-1}a_n^{-\alpha-1},\end{aligned}$$

while by (4.4)

$$\overline{H}(x) \geq \overline{F}^2(x/2) \geq \overline{F}^2(3 \cdot 2^{t-2}a_n) \geq C^2 a_n^{-2\alpha}\Big(1 - \Big(2 \cdot (3 \cdot 4^{-1})^{1/t} - 1\Big)^b\Big)^2.$$

Since $\alpha < 1$, $\sup_{x \in J_{n,1}} h(x)/H(x) \to 0$ as $n \to \infty$.

In the case (ii), representations (4.3) and (4.7) imply that

$$\begin{aligned} h(x) &= 2Cbt^{-1}a_n^{-\alpha-1/t} \int_{2^{-1}x}^{2^t a_n} y^{1/t-1}\Big((ya_n^{-1})^{1/t} - 1\Big)^{b-1} f(x-y)dy \\ &\leq 2Cbt^{-1}a_n^{-\alpha-1} \int_{x-2^t a_n}^{x/2} f(y)dy \\ &\leq 2Cbt^{-1}a_n^{-\alpha-1}\overline{F}(2^{t-1}a_n) \leq 2C^2bt^{-1}a_n^{-2\alpha-1}, \end{aligned}$$

and by (4.4) we get that

$$\overline{H}(x) \geq \overline{F}(x) \geq Ca_n^{-\alpha-1}.$$

Then again $\sup_{x \in J_{n,2}} h(x)/H(x) \to 0$ as $n \to \infty$.

We may conclude that (4.8) holds, therefore $F^{*2} \in \mathcal{L}$.

In order to prove $F^{*2} \in \mathcal{OS}$, we only need to show that

$$T(x) = \int_{x/2}^{x} \overline{H}(x-y)h(y)dy = O(\overline{H}(x)). \tag{4.9}$$

It is clear that $T(x) > 0$ if and only if $a_n + a_0 < x < 2^{t+2}a_n$, for $n = 0, 1, \cdots$.

By (4.3) and (4.7), it is easy to see that, for $n = 0, 1, \dots$, if $x \in [a_n + a_0, 2^t a_n)$, then

$$h(x) = 2\int_{a_n}^{x} f(x-y)f(y)dy \leq 2Cbt^{-1}a_n^{-\alpha-1}, \tag{4.10}$$

and if $x \in [2^t a_n, 2^{t+1}a_n)$, then

$$h(x) = 2\int_{x/2}^{2^t a_n} f(x-y)f(y)dy \leq 2Cbt^{-1}a_n^{-\alpha-1}\overline{F}(x - 2^t a_n). \tag{4.11}$$

Then we estimate $T(x)$ separately in three cases: (i) $x \in [a_n + a_0, 3 \cdot 2^{t-1}a_n)$, (ii) $x \in [3 \cdot 2^{t-1}a_n, 2^{t+1}a_n)$ and (iii) $x \in [2^{t+1}a_n, 2^{t+2}a_n)$ for $n = 0, 1, \ldots$.

In the case (i), representations (4.4), (4.10) and (4.11) lead to

$$\begin{aligned} T(x)/\overline{H}(x) &\leq \max_{y \in [x/2, x]}\{h(y)\} \int_{x/2}^{x} \overline{H}(x-y)dy/\overline{F}^2(2^{-1}x) \\ &\leq 2Cbt^{-1}a_n^{-\alpha-1} \int_0^{3 \cdot 2^{t-2}a_n} \overline{H}(y)dy/\overline{F}^2(3 \cdot 2^{t-2}a_n) < \infty. \end{aligned}$$

In the case (ii), representations (4.4), (4.10) and (4.11) imply that

$$\begin{aligned}
T(x)/\overline{H}(x) &\le \Big(\int_{x/2}^{2^t a_n} + \int_{2^t a_n}^{x}\Big)\overline{H}(x-y)h(y)dy\Big/\Big(\overline{F}(x) + \int_{x/2}^{x}\overline{F}(x-y)F(dy)\Big) \\
&\lesssim 2bt^{-1}\int_{x/2}^{2^t a_n}\overline{H}(x-y)dy\Big/\Big(1+\int_{x/2}^{2^t a_n}\overline{F}(x-y)dy\Big) \\
&\qquad +2bt^{-1}\int_{2^t a_n}^{x}\overline{H}(x-y)\overline{F}(y-2^t a_n)dy \\
&\le 4bt^{-1}\int_{x-2^t a_n}^{x/2}\Big(\overline{F}(y)+\int_{y/2}^{y}\overline{F}(y-z)F(dz)\Big)dy\Big/\Big(1+\int_{x-2^t a_n}^{x/2}\overline{F}(y)dy\Big) \\
&\qquad +2bt^{-1}\Big(\int_0^{x/2-2^{t-1}a_n} + \int_{x/2-2^{t-1}a_n}^{x-2^t a_n}\Big)\overline{H}(y)\overline{F}(x-2^t a_n - y)dy \\
&\le 4bt^{-1}\Big(1+2Cbt^{-1}a_n^{-\alpha-1}(2^t a_n - 2^{-1}x)\int_0^{4^{-1}x}\overline{F}(z)dz\Big) \\
&\qquad +2bt^{-1}\Big(\overline{F}(2^{t-2}a_n)\int_0^{2^{t-1}a_n}\overline{H}(y)dy + \overline{H}(2^{t-2}a_n)\int_0^{2^{t-1}a_n}\overline{F}(y)dy\Big) \\
&\le 4bt^{-1} + O(a_n^{2\alpha-1}) < \infty.
\end{aligned}$$

Recall that, for two positive functions f and g, notation $f(x) \lesssim g(x)$ means that $\limsup_{x\to\infty} f(x)/g(x) \le 1$.

In the case (iii), representations (4.4) and (4.11) show that

$$\begin{aligned}
T(x)/\overline{H}(x) &= \int_{x/2}^{2^{t+1}a_n}\overline{H}(x-y)h(y)dy/\overline{H}(x) \\
&\lesssim bt^{-1}\int_{2^t a_n}^{2^{t+1}a_n}\overline{H}(2^{t+1}a_n - y)\overline{F}(y-2^t a_n)dy \\
&= bt^{-1}\Big(\int_0^{2^{t-1}a_n} + \int_{2^{t-1}a_n}^{2^t a_n}\Big)\overline{H}(y)\overline{F}(2^t a_n - y)dy \\
&\le bt^{-1}\overline{F}(2^{t-1}a_n)\int_0^{2^{t-1}a_n}\overline{H}(y)dy + bt^{-1}\overline{H}(2^{t-1}a_n)\int_0^{2^{t-1}a_n}\overline{F}(y)dy \\
&= O(a_n^{2\alpha-1}) < \infty.
\end{aligned}$$

We may conclude that (4.9) holds, therefore $F^{*2} \in \mathcal{OS}$.

Finally, since $F \notin \mathcal{L}$ and $F^{*2} \in \mathcal{L}$, Proposition 2.2 leads to the conclusion that $F \notin \mathcal{OS}$. □

Proof of (2). Since $F \notin \mathcal{L}$, we have $F^{*\tau} \notin \mathcal{S}$, by [13]. Under condition (2.13), $F^{*\tau} \in \mathcal{L}$ follows from $F^{*2} \in \mathcal{L}$ and part (3) of Theorem 2.1.

Under condition (2.14) with any fixed $0 < \varepsilon < 1$ and $M = M(\varepsilon) \ge n$ large enough, Corollary 2.1 implies that

$$(1-\varepsilon)\overline{F^{*\tau}}(x-1) \le \sum_{n=1}^{M} p_n\overline{F^{*n}}(x-1) \le (1+\varepsilon)\sum_{n=1}^{M} p_n\overline{F^{*n}}(x) \le (1+\varepsilon)\overline{F^{*\tau}}(x),$$

for x large enough. Since $\varepsilon > 0$ is arbitrary, we get $F^{*\tau} \in \mathcal{L}$.

Further, we prove that $F^{*\tau} \in \mathcal{OS}$ under condition (2.14). Without loss of generality, we may assume that $p_M > 0$. By $F^{*2} \in \mathcal{OS}$ and Proposition 2.6 in [29], we have $F^{*M} \in \mathcal{OS}$. Further, by (2.14), we have

$$(1-\varepsilon)\overline{F^{*\tau}}(x) \;\; \le \;\; \sum_{i=1}^{M} p_i \overline{F^{*i}}(x) = O(\overline{F^{*M}}(x)).$$

On the other hand, relation $\overline{F^{*M}}(x) = O(\overline{F^{*\tau}}(x))$ is clear. Therefore, $F^{*\tau} \in \mathcal{OS}$ follows from $F^{*M} \in \mathcal{OS}$.

Next, we prove (2.15). Recall that all distributions from the class $\mathcal{F}_1(0)$ are supported by the positive half-line. Since $\mathbf{E}X_1 = \infty$ and $\mathbf{E}\tau < \infty$, Theorem 1 of Denisov et al. [7] implies the first equality in (2.15) (see also [28], for a particular case of power tails). Then the first inequality in (2.15) follows, say, by [18]. Further, since $\tau \ge 2[\tau/2]$ a.s. (here $[x]$ is the integer part of x), the second inequality is straightforward:

$$\begin{aligned}
\liminf \overline{F^{*\tau}}(x)/\overline{F^{*2}}(x) \;\; &\ge \;\; \liminf \overline{F^{*2[\tau/2]}}(x)/\overline{F^{*2}}(x) \\
&= \;\; \liminf \sum_{m=1}^{\infty} (p_{2m} + p_{2m+1})\overline{F^{*2m}}(x)/\overline{F^{*2}}(x) \\
&\ge \;\; \sum_{m=1}^{\infty} (p_{2m} + p_{2m+1}) \liminf \overline{F^{*2m}}(x)/\overline{F^{*2}}(x) = \sum_{m=1}^{\infty} (p_{2m} + p_{2m+1})m,
\end{aligned}$$

where the last equality follows again by [7].

Finally, we prove (2.17). Since $F \notin \mathcal{OS}$ and F is supported by the positive half-line, the last equality in (2.17) follows. By $F^{*2} \in \mathcal{L} \cap \mathcal{OS}$ and the corresponding Kesten's type inequality, see Lemma 5 in [36], for any $\varepsilon > 0$ there is a constant $K = K(\varepsilon) > 0$ such that, for all $n \ge 1$ and $x \ge 0$,

$$\overline{F^{*2m}}(x)/\overline{F^{*2}}(x) \le K(C^*(F^{*2}) - 1 + \varepsilon)^m.$$

Further, by Lemma 4 or Remark 2 in [36], for all $m \ge 1$,

$$\limsup \overline{F^{*2m}}(x)/\overline{F^{*2}}(x) \le m(C^*(F^{*2}) - 1)^{m-1}.$$

Thus, by condition (2.18) with $n = 2$ and the dominated convergence theorem, we obtain the first inequality in (2.17):

$$\begin{aligned}
\limsup \overline{F^{*\tau}}(x)/\overline{F^{*2}}(x) \;\; &\le \;\; \limsup \overline{F^{*2[(\tau+1)/2]}}(x)/\overline{F^{*2}}(x) \\
&= \;\; \limsup \sum_{m=1}^{\infty} (p_{2m-1} + p_{2m})\overline{F^{*2m}}(x)/\overline{F^{*2}}(x) \\
&\le \;\; \sum_{m=1}^{\infty} m(p_{2m-1} + p_{2m})(C^*(F^{*2}) - 1)^{m-1} < \infty.
\end{aligned}$$

$\square$

Proof of (3). Let H be an infinitely divisible distribution on the positive half-line. The Laplace transform of H is given by

$$\int_0^\infty \exp\{-\lambda y\} H(dy) = \exp\{-a\lambda - \int_0^\infty (1 - e^{\lambda y})\upsilon(dy)\}$$

where $a \geq 0$ is a constant and the Lévy measure υ is a Borel measure supported by $(0, \infty)$ with the properties $\mu = \upsilon((1, \infty)) < \infty$ and $\int_0^1 y\upsilon(dy) < \infty$ – see, for example, Feller [16], page 450. Let $F(x) = \mu^{-1}\upsilon(x) = \mu^{-1}\upsilon((0, x])$ for $x > 0$.

It is well-known that the distribution H admits the representation $H = H^{(1)} * H^{(2)}$, where $\overline{H^{(1)}}(x) = O(e^{-\beta x})$ for some $\beta > 0$ and

$$H^{(2)}(x) = e^{-\mu} \sum_{n=0}^{\infty} \frac{\mu^n}{n!} F^{*n}(x).$$

Let a random variable τ have a Poisson distribution, $p_n = e^{-\mu}\frac{\mu^n}{n!}$ for $n = 0, 1, \cdots$. Take a distribution $F \in \mathcal{F}_1(0)$. Since a Poisson distribution has unbounded support and is light-tailed, condition (2.18) is fulfilled and $H^2 \in \mathcal{L} \cap \mathcal{OS}$, by part (2) of Theorem 2.1. Since $H^{(1)}$ is light-tailed, we have $\overline{H}(x) \sim \overline{H^{(2)}}(x)$, by Property 2. Then, clearly, $H \in \mathcal{L} \cap \mathcal{OS}$. Since distribution G is Poisson, condition (2.17) holds. Finally, since $F \notin \mathcal{S}$, Theorem 1 of Embrechts et al. [13] leads to $H \notin \mathcal{S}$. □

Proof of Proposition 2.2. By Theorem 3.1 (b) of Embrechts and Goldie [13], we need to prove the implication $\Leftarrow$ only. By $F^{*2} \in \mathcal{L}$ and Corollary 2.1 (2), we know that $G_2 =: pF + qF^{*2} \in \mathcal{L}$ for any $p + q = 1$ and $0 < q < 1$. Further, since $F \in \mathcal{OS}$, we have $\overline{G_2}(x) = O(\overline{F}(x))$. Therefore, $F \in \mathcal{L}$ follows from Lemma 2.4 of Yu et al. [35]. □

Proof of Lemma 2.1. By $F^{*n} \in \mathcal{L} \cap \mathcal{OS}$ and Lemma 5 of Yu and Wang [36], for any $0 < \varepsilon_0 < 1$, there exists a constant $K = K(\varepsilon_0) > 0$ such that, for all $x > 0$ ang $m \geq 1$,

$$\overline{F^{*mn}}(x) \leq K(C^*(F^{*n}) - 1 + \varepsilon_0)^m \overline{F^{*n}}(x).$$

Then, by (2.18), for any $0 < \varepsilon < 1$, there exists an integer $M_0 = M_0(\varepsilon) > 1$ large enough such that

$$\sum_{k=(M_0-1)n}^{\infty} p_k \overline{F^{*k}}(x) \leq \sum_{m=M}^{\infty} \Big(\sum_{k=(m-1)n+1}^{mn} p_k \Big) \overline{F^{*mn}}(x) \leq \varepsilon \overline{F^{*\tau}}(x).$$

Take $M = (M_0 - 1)n$, then (2.14) holds. □

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5 Appendix

5.1 On condition (2.2)

The following two examples show the feasibility of condition (2.2).

Example 5.1. *Take a distribution G_1 given by*

$$\overline{G_1}(x) = \mathbf{I}(x < 0) + e^{-\sqrt{x}}\mathbf{I}(x \geq 0).$$

Xu et al. [33] in their Example 2.1 introduce a distribution F_1 on the positive half-line such that, for $x \in (-\infty, \infty)$,

$$\overline{F_1}(x) \;\; = \;\; \overline{G}_1(x)\mathbf{I}(x < x_1) + \sum_{n=1}^{\infty} \Big(\overline{G}_1(x_n)\mathbf{I}(x_n \le x < y_n) + \overline{G}_1(x)\mathbf{I}(y_n \le x < x_{n+1}) \Big),$$

where $\{x_n, n \ge 1\}$ and $\{y_n, n \ge 1\}$ are two sequences of positive constants satisfying $x_n < y_n < x_{n+1}$ and $\overline{G}_1(x_n) = 2\overline{G}_1(y_n)$, $n \ge 1$. One can easily verify that $F_1 \in \mathcal{OL} \setminus \mathcal{L}$ and $\overline{F_1}(x) \asymp \overline{G_1}(x)$, that is $0 < \liminf \overline{G}_1(x)/\overline{F}_1(x) \le \limsup \overline{G}_1(x)/\overline{F}_1(x) < \infty$. Further, take a distribution F_2 such that

$$\overline{F_2}(x) = \mathbf{I}(x < 0) + \overline{G_1}(x)\mathbf{I}(x \ge 0)/\log(x + 2).$$

Clearly, $F_2 \in \mathcal{S} \subset \mathcal{L}$, $\overline{F_2}(x) = o(\overline{F_1}(x))$ and condition (2.2) holds. Then Remark 2.1 or, equivalently, part (1b) of Theorem 2.1 imply that $F_1 * F_2 \in \mathcal{L}$. □

Example 5.2. *Assume $\overline{F_2}(x) = x^{-\alpha}$ for $x \ge 1$, where $\alpha > 0$. Let $1 > \varepsilon_n \downarrow 0$ be any decreasing sequence. Given two sequences $\{a_n, n \ge 1\}$ and $\{b_n, n \ge 1\}$ such that*

$$1 = a_1 < b_1 < \ldots < a_n < b_n < a_{n+1} < b_{n+1} < \ldots,$$

we let

$$\overline{F_1}(x) = \mathbf{I}(x < a_1) + \sum_{n=1}^{\infty} c_n \mathbf{I}(x \in [a_n, b_n]) + \sum_{n=1}^{\infty} d_n x^{-2\alpha} \mathbf{I}(x \in (b_n, a_{n+1})).$$

Here $c_1 = 1$, $d_n = c_n b_n^{2\alpha}$ and $c_{n+1} = d_n a_{n+1}^{-2\alpha} \varepsilon_n$. Then we may determine sequences $\{a_n, n \ge 1\}$ and $\{b_n, n \ge 1\}$ recursively in such a way that

$$\frac{\overline{F_1}(b_n)}{\overline{F_2}(b_n)} = c_n b_n^{\alpha} = 2^n \to \infty \quad \textit{and} \quad \frac{\overline{F_1}(a_n - 0)}{\overline{F_2}(a_n)} = d_{n-1} a_n^{-\alpha} = 2^{-n+1} \to 0, \qquad \textit{as} \;\; n \to \infty. \qquad (5.1)$$

Informally, we proceed as follows. Let $a_1 = c_1 = 1$ and choose b_1 such that $\overline{F_2}(b_1) = 1/2$, then $d_1 = b_1^{2\alpha}$. Then choose a_2 such that $d_1 a_2^{-2\alpha} = 2^{-1} a_2^{-\alpha}$ and then $c_2 = d_1 a_2^{-2\alpha} \varepsilon_1$. By the induction argument, given a_n and c_n, we keep $F_1(x)$ constant in the interval $[a_n, b_n]$. Since $\overline{G}$ decreases to 0 continuously, we may choose b_n so large that the first equation in (5.1) *holds. Then, by the symmetric argument, we may choose a_{n+1} so large that the second equation in* (5.1) *holds, with a_{n+1} in place of a_n.*

One can see that $F_1 \notin \mathcal{OL}$. However, condition (2.2) is satisfied, thus $F = F_1 * F_2 \in \mathcal{L}$, by Remark 2.1 or part (1b) of Theorem 2.1. □

5.2 Sketch of the Proof of Proposition 2.1

The proof mostly follows the lines of the proof of Theorem 2.2, so we provide its sketch only, and also a complete proof of the last new statement.

We first analyse the distribution F of the random variable ξ and its density f. Clearly, $\eta^{1/t} \le \xi \le (2\eta)^{1/t}$. Since

$$\mathbf{E}\eta^{s/t} = C \sum_{n=0}^{\infty} a_n^{s/t - \alpha} < \infty$$

if and only if $s < t\alpha$, the same holds for ξ, and the distribution F is heavy-tailed with infinite mean. Next, for all x, we get

$$\begin{aligned}
& \mathbf{P}(a_n^{1/t} < \xi \le x)\mathbf{I}(x \in [a_n^{1/t}, a_{n+1}^{1/t})) \\
= \; & C a_n^{-\alpha}\left(a_n^{-1}x^t - 1\right)\mathbf{I}(x \in [a_n^{1/t}, (2a_n)^{1/t})) + C a_n^{-\alpha}\mathbf{I}(x \in [(2a_n)^{1/t}, a_{n+1}^{1/t})),
\end{aligned}$$

then

$$f(x) = Ct\sum_{n=0}^{\infty} x^{t-1} a_n^{-\alpha-1}\mathbf{I}(x \in [a_n^{1/t}, (2a_n)^{1/t}))$$

and

$$\begin{aligned}
\overline{F}(x) \; = \; & \mathbf{I}(x < a_0) + C\sum_{n=0}^{\infty}\Big(\big(\sum_{i=n}^{\infty} a_i^{-\alpha} - a_n^{-\alpha}(a_n^{-1}x^t - 1)\big)\mathbf{I}(x \in [a_n^{t^{-1}}, (2a_n)^{t^{-1}})) \\
& + \sum_{i=n+1}^{\infty} a_i^{-\alpha}\mathbf{I}(x \in [(2a_n)^{1/t}, a_{n+1}^{1/t}))\Big).
\end{aligned}$$

Then we follow the lines of the proof of Theorem 2.2 to show that $F \notin \mathcal{OL}$ and that $F^{*2} \in \mathcal{L}$, by considering again the three cases.

Then we come to the proof of the two last statements: $F^{*2} \in \mathcal{OS}$ if $\alpha \in [1/2, 1/t)$, and $F^{*2} \notin \mathcal{OS}$ if $\alpha \in (1 - 1/t, 1/2)$. The proof in the case $\alpha \in [1/2, 1/t)$ is again analogous to the corresponding part of the proof of Theorem 2.2, so we turn to the proof of the latter result.

Let again $H = F^{*2}$, h be the density of H, and $T(x) = \int_{x/2}^{x}\overline{H}(x-y)h(y)dy$. For $\alpha \in (1 - 1/t, 1/2)$, we have

$$\begin{aligned}
& T(2(2a_n)^{1/t})/\overline{H}(2(2a_n)^{1/t}) \\
\ge \; & \int_{(2^{1/t}+1)a_n^{1/t}}^{2(2a_n)^{1/t}} 2\overline{H}(2(2a_n)^{1/t} - y)\int_{y/2}^{(2a_n)^{1/t}} f(y-z)f(z)dzdy/\overline{H}(2(2a_n)^{1/t}) \\
\gtrsim \; & t a_n^{1-1/t}\int_{(2^{1/t}+1)a_n^{1/t}}^{2(2a_n)^{1/t}} \overline{H}(2(2a_n)^{1/t} - y)(\overline{F}(y - (2a_n)^{1/t}) - \overline{F}(y/2))dy \\
\ge \; & Ct a_n^{-\alpha-1/t}\int_{(2^{1/t}+1)a_n^{1/t}}^{2(2a_n)^{1/t}} \overline{H}(2(2a_n)^{1/t} - y)((2a_n)^{1/t} - y/2)(y/2)^{t-1}dy \\
\ge \; & C2^{-1}t a_n^{1-\alpha-2/t}\int_0^{(2^{1/t}-1)a_n^{1/t}} \overline{H}(y)ydy \to \infty, \quad n \to \infty.
\end{aligned}$$

Here notation $f(x) \gtrsim g(x)$ is equivalent to $g(x) \lesssim f(x)$ and means that $\liminf_{x\to\infty} f(x)/g(x) \ge 1$. Thus, $F^{*2} \notin \mathcal{OS}$.

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